1.5

I've Created a Monster, $m(x)$

Analyzing Graphs to Build New Functions

LEARNING GOALS

In this lesson, you will:

- Model operations on functions graphically.
- Sketch the graph of the sum, difference, and product of two functions on a coordinate plane.
- Predict and verify the graphical behavior of functions.
- Build functions graphically.
- Predict and verify the behavior of functions using a table of values.
- Build functions using a table of values.

KEY TERM

- Zero Product Property
- polynomial
- degree

In 1818 Mary Shelley wrote the science fiction novel *Frankenstein*. It is the tale of Dr. Victor Frankenstein, a scientist who dreams of creating life. He accomplishes this dream by using old body parts and electricity. Unfortunately, he creates a monster! Horrified and filled with regret, Victor decides that he must end the life that he created. His monster has other plans, though. He is lonely and wants Victor to create a woman to keep him company in this cruel world! Crime, drama, and vengeance follow as the creator struggles with his creation.

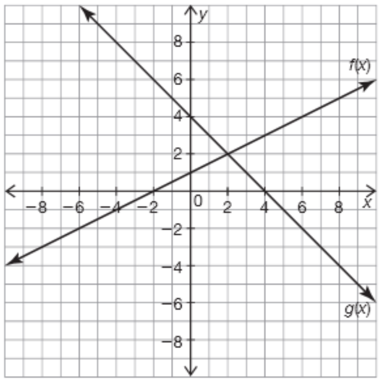
Frankenstein laid the foundation for many of the horror and science fiction movies that you see today. While Mary Shelley's novel is a literary classic for how it tackles deep issues such as the meaning of life and the ethics of creation, it is also good old-fashioned, scary fun. Do you enjoy scary movies? If so, do you think any of your favorites may have been influenced by this classic tale?

1

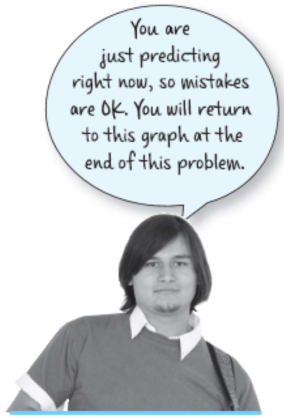
PROBLEM 1 It's Moving . . . It's Alive!

In the problem, *You're So Square*, you added the functions $w(n)$ and $g(n)$ algebraically to create a new function $t(n)$. Manipulating algebraic representations is a common method for building new functions. However, you can also build new functions graphically. Let's consider two graphs of functions on a coordinate plane and what happens when you add, subtract, or multiply the output values of each.

1. Analyze the graphs of $f(x)$ and $g(x)$.



a. Predict the function family of $m(x)$ if $m(x) = f(x) + g(x)$. Explain your reasoning.



b. Predict and sketch the graph of $m(x)$.

c. Explain how you predicted the location of $m(x)$.

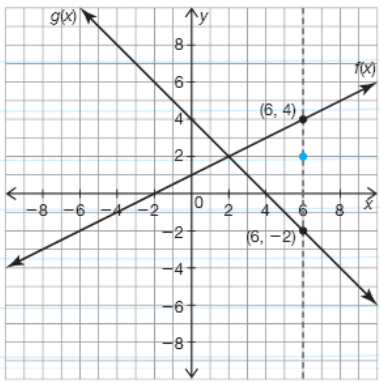
A graph of a function is a set of an infinite number of points. When you add two functions you are adding the output values for each input value. Given two functions, $f(x)$ and $g(x)$, on a coordinate plane, you can graphically add these functions to produce a new function, $m(x)$. To get started, let's consider what happens when you add $f(x)$ and $g(x)$ at a single point.

Let's add the output values for $f(x)$ and $g(x)$ at $x = 6$ to determine $(6, m(6))$.

$m(6) = f(6) + g(6)$
$m(6) = 4 + -2$
$m(6) = 2$

The point on the new function $m(x)$ is $(6, 2)$.

Other points on the graph of $m(x)$ can be determined in a similar way.



2. Analyze the addition of the output values for the input value $x = 6$ in the worked example.

   a. How is this process similar to adding integers on a number line?

   b. Why is the point $(6, m(6))$ closer to $f(x)$ than $g(x)$?

   c. Why did the input value of 6 stay the same while the output values changed?

   d. Choose a different input value. Add the output values for $f(x)$ and $g(x)$ to determine a new point on the graph of $m(x)$.

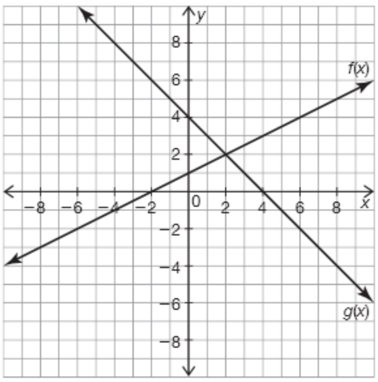
1

Now, let's consider what happens when you add $f(x)$ and $g(x)$ at a few other points. The properties you use in integer operations also extend to operations on the graphs of functions. Recall the integer properties shown in the table.

Property	Definition	Integer Example
Commutative Property over Addition	The commutative property states that the order in which the terms are added does not change the sum. In other words $a + b = b + a$.	$35 + 43 = 43 + 35$
Additive Inverse	The additive inverse of a number is the number such that the sum of the given number and its additive inverse is 0.	The numbers -5 and 5 are additive inverses because $-5 + 5 = 0$.
Additive Identity	The additive identity is 0 because any number added to 0 is equal to itself.	$5 + 0 = 5$

3. Extend the integer properties from the table to operations on the graphs of functions.

a. Use two output values from functions $f(x)$ and $g(x)$ to demonstrate the commutative property over addition for functions.



b. Determine output values for $f(x)$ and $g(x)$ that demonstrate the Additive Inverse Property. Show that they are additive inverses algebraically and graphically.

c. Determine output values for $f(x)$ and $g(x)$ that demonstrate the Additive Identity Property. Show that they are additive identities algebraically and graphically.

1

4. Ari and Will disagree over the location of $(2, m(2))$ when the output values of the functions $f(x)$ and $g(x)$ are added.

Ari

$g(2) + f(2) = (2, m(2))$

$(2, 2) + (2, 2) = (2, 4)$

The location of $(2, m(2))$ is $(2, 4)$.

The two points are at the intersection. Adding the output values of the two points equals $(2, 2 + 2)$.

Will

$(2, 2) + 0 = (2, 2)$

The location of $(2, m(2))$ is $(2, 2)$.

The lines intersect at one point.

A point plus zero is itself.

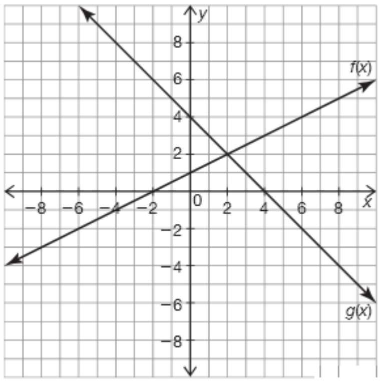
Who is correct? Explain your reasoning.

1

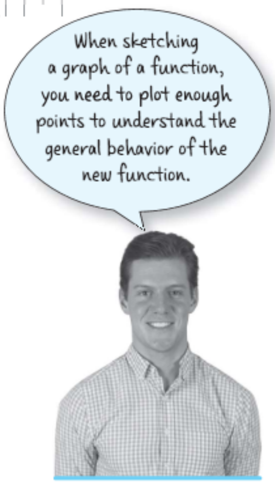
When performing operations on two graphs, it isn't practical to consider all sets of ordered pairs. The process is much more efficient if you use key points. Some of the points considered in this problem, such as intercepts, zeros, and intersection points, are good examples of key points.

5. Sketch the graph of $m(x) = f(x) + g(x)$.

   a. Circle key points of the graphs of $f(x)$ and $g(x)$.



   b. Draw dashed vertical lines through your key points.



   c. Add the corresponding y-values of $f(x)$ and $g(x)$ on each dashed vertical line to determine points on $m(x)$. Then sketch the graph of $m(x)$. Show or explain your work.

   d. Verify your graph of $m(x)$ using one or more pairs of points that are not key points.

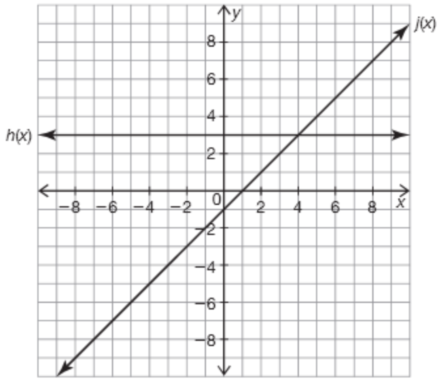
   e. Compare the function you graphed in this question with the prediction you made in Question 1. Describe any errors you may have made in your prediction.

PROBLEM 2 Keep On Keeping On

1

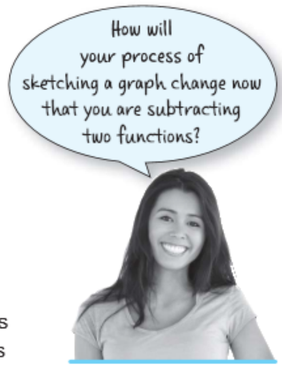
Let's consider operations on different types of graphs. Let's look at a linear function and a constant function.

1. Analyze the graphs of $j(x)$ and $h(x)$.



a. Predict the function family of $c(x)$ if $c(x) = j(x) + h(x)$. Then sketch the graph of $c(x)$.

b. Describe the relationship between original functions and $c(x)$. Explain the relationship between the functions in terms of their graphical and algebraic representations.



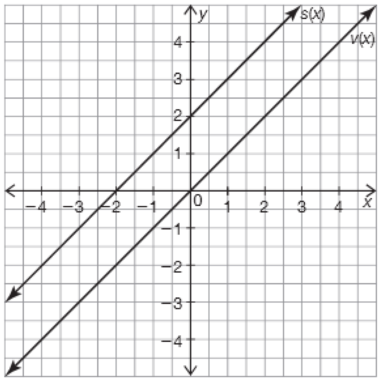
c. Predict the function family of $n(x)$ if $n(x) = j(x) - h(x)$. Then sketch the graph of $n(x)$.

d. Describe the relationship between the original functions and $n(x)$. Explain the relationship between the functions in terms of their graphical and algebraic representations.

1

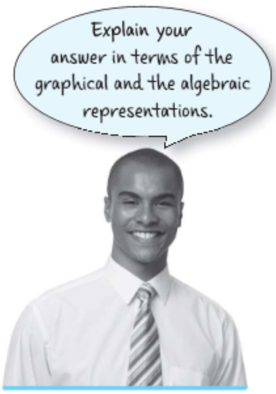
Now let's look at what happens when you add and subtract the outputs of two parallel lines.

2. Analyze the graphs of $s(x)$ and $v(x)$.

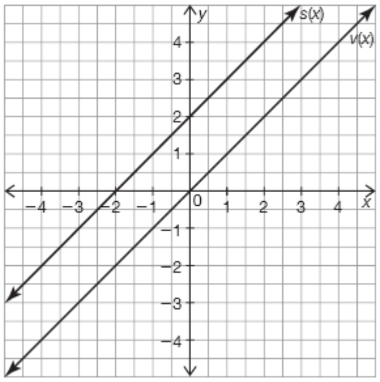


a. Sketch the graph of $w(x) = s(x) + v(x)$.

b. Describe the shape of $w(x)$ compared to $s(x)$ and $v(x)$. Explain why adding the output values changes the shape of the new graph in this way.

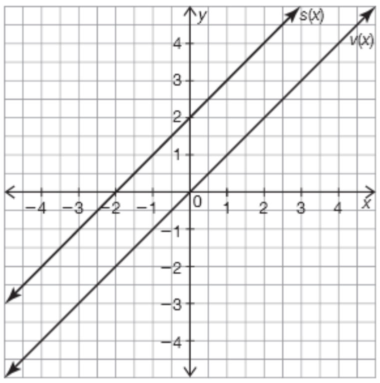


c. Sketch the graph of $m(x)$ if $m(x) = s(x) - v(x)$.



1

d. Sketch the graph of $n(x)$ if $n(x) = v(x) - s(x)$.

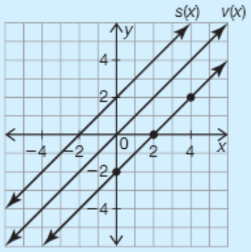


e. Describe the shape of the graph when you subtract $s(x)$ and $v(x)$. Will subtracting the output values of any two parallel lines have this same result? Explain your reasoning.

1

3. Mrs. Webb asked her students to determine $v(x) - s(x)$. Erik's and Lily's work is shown.

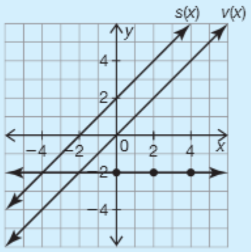
Erik



v(x)	t(x)	Differences
0	−2	−2
2	0	−2
4	2	−2

The new graph is located 2 units below v(x). I know this is correct because each point has a difference of −2

Lily



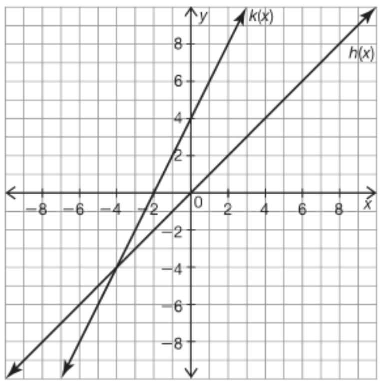
The new graph is located at $y = -2$. I know this is correct because I subtracted several points and the y-value was always −2.

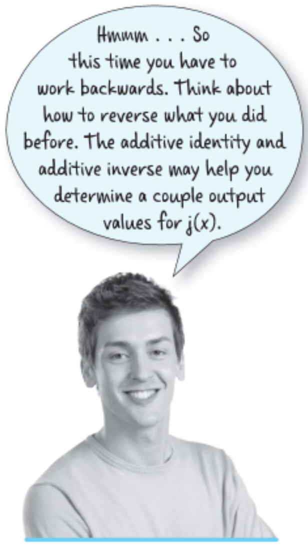
Who's correct? Explain why one graph is correct and the error made to create the other graph.

Now, let's work backwards.

4. Analyze the graphs of $h(x)$ and $k(x)$.

a. Draw the function $j(x)$ with outputs such that $h(x) + j(x) = k(x)$.





b. Complete the table of values to verify that $h(x) + j(x) = k(x)$.

x	h(x)	j(x)	k(x) = h(x) + j(x)
−2			
−1			
0			
1			
2			

c. Describe examples of the additive inverse and additive identity properties for output values in this problem.

d. Use the graph or table of values to determine the algebraic expressions for $h(x)$, $j(x)$, and $k(x)$. Algebraically show that $h(x) + j(x)$ is equivalent to $k(x)$.

e. How can you determine from the graph, the table of values, and the algebraic expressions that the functions $h(x)$, $j(x)$, and $k(x)$ are all linear?

graph:

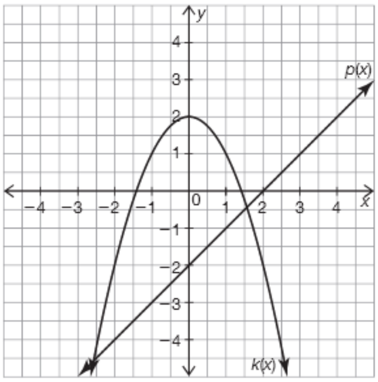
table:

equation:

f. Do you think adding two linear functions will always result in another linear function? Explain your reasoning.

So far, you have only considered two linear functions. Now let's explore a linear function and a quadratic function.

5. Analyze the graphs of $k(x)$ and $p(x)$.



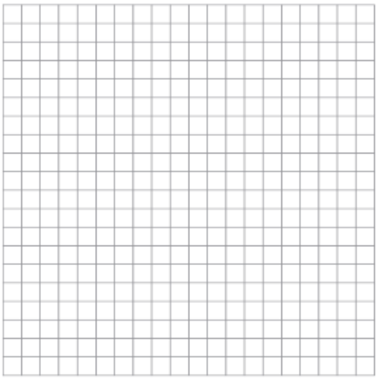
a. Predict the function family of $a(x)$ if $a(x) = k(x) + p(x)$. Explain your prediction.

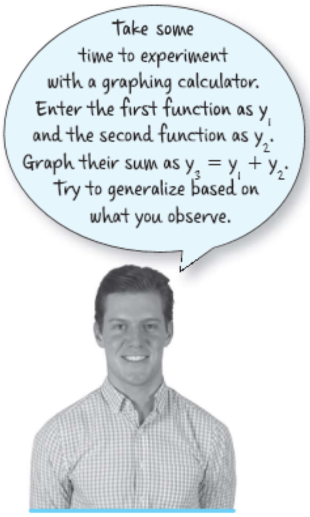
b. Sketch the graph of function $a(x)$. Show or explain your work.

c. Do you think adding a linear function and a quadratic function will always result in a quadratic function? Explain your reasoning in terms of the algebraic and graphical representations of the functions.

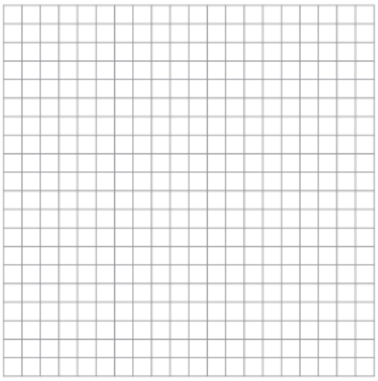
6. Draw the graphs that meet the criteria provided.

   a. Sketch the graph of two different functions whose sum is a parabola opening up. What conclusions can you make about the two functions?



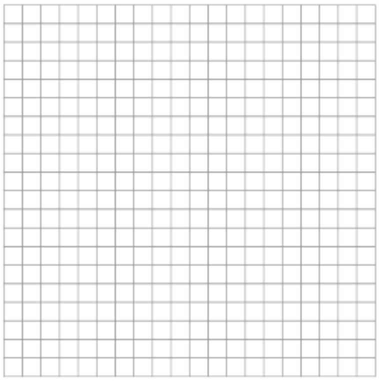


   b. Sketch the graph of two functions whose sum is the horizontal line $y = 0$. What conclusions can you make about the two functions?



1

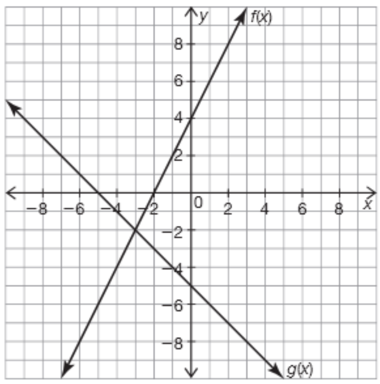
c. Sketch the graph of two functions whose sum is not a function. What conclusions can you make about the two functions?

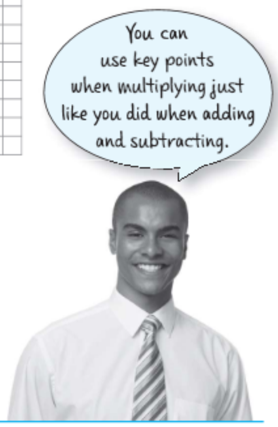


PROBLEM 3 They're Multiplying!!

Just as you added and subtracted functions in the previous problems, you can also build functions through multiplication.

1. Analyze the graphs of $f(x)$ and $g(x)$.





a. Predict the function family of $h(x)$ if $h(x) = f(x) \cdot g(x)$. Explain your reasoning.

b. Sketch the graph of $h(x)$. Show or explain your work.

c. Describe the differences between the graphs of $f(x)$ and $g(x)$ and the graph of $h(x)$.

d. Was your prediction in part (a) correct? What was the same/different after you multiplied the output values of key points?

2. You can analyze a table of values to determine the graphical behavior of functions.

a. Complete the table of values for $h(x) = f(x) \cdot g(x)$.

x	$f(x)$	$g(x)$	$h(x)$
−7	−10	2	
−6	−8	1	
−5	−6	0	
−4	−4	−1	
−3	−2	−2	
−2	0	−3	
−1	2	−4	
0	4	−5	

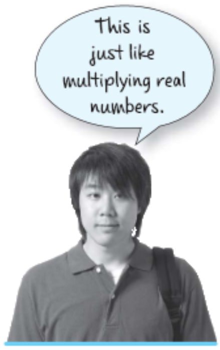
b. What patterns do you notice in the table?

c. Analyze the first and second differences for each function. How do you know $f(x)$ and $g(x)$ are linear but $h(x)$ is not?

1

3. Consider the sign of the output values for each function in the table.

   a. For which input values are the output values of $h(x)$ negative? For which input values are the output values of $h(x)$ positive?



   b. How does the sign of the output values of $f(x)$ and $g(x)$ determine the sign of the output values of $h(x)$?

4. Consider the x-intercepts for $f(x)$, $g(x)$ and $h(x)$.

   a. Identify the x-intercepts for each function.

      $f(x)$:

      $g(x)$:

      $h(x)$:

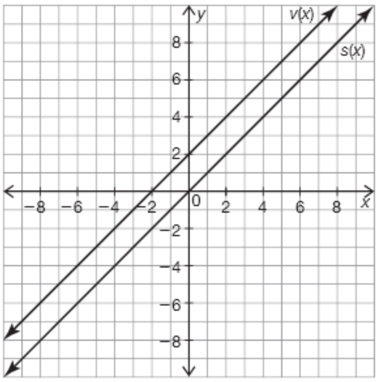
   b. What pattern do you notice in the x-intercepts?

The **Zero Product Property** states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero.

   c. How does the Zero Product Property relate to the x-intercepts of the three functions?

1

5. Analyze the graphs of $s(x)$ and $v(x)$.



a. Sketch the graph of $p(x)$ if $p(x) = s(x) \cdot v(x)$.

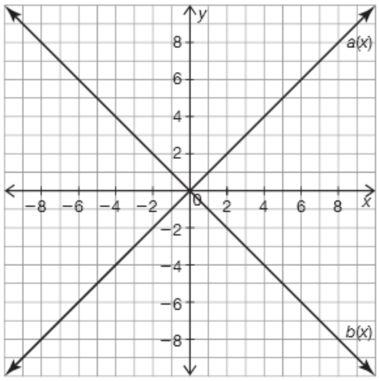
b. Identify the x-intercepts of $p(x)$. Explain the relationship between the x-intercepts of $p(x)$ and the x-intercepts of $s(x)$ and $v(x)$.

c. Identify the vertex of $p(x)$. What is the relationship between the vertex of $p(x)$ and the functions $s(x)$ and $v(x)$?

d. In Problem 2 of this lesson, you added the functions $s(x)$ and $v(x)$ to create function $w(x)$. How is multiplication the same? How is it different?

1

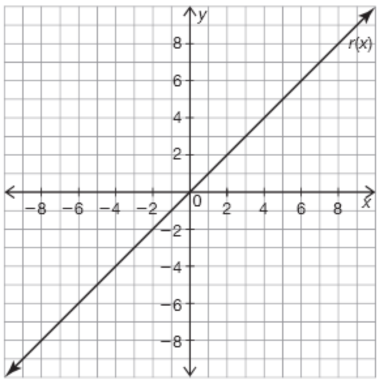
6. Analyze the graphs of $a(x)$ and $b(x)$.



a. Sketch the graph of $c(x)$ if $c(x) = a(x) \cdot b(x)$.

b. Describe the shape of $c(x)$.

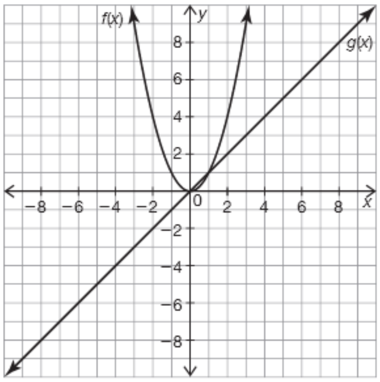
7. Analyze the graph of $r(x)$.



a. Sketch the graph of $d(x)$ if $d(x) = r(x) \cdot r(x)$.

b. Describe the shape of $d(x)$.

8. Analyze the graphs of $f(x)$ and $g(x)$.



a. Sketch the graph of $m(x)$ if $m(x) = f(x) \cdot g(x)$.

b. Describe the shape of $m(x)$.

c. Do you think multiplying a quadratic function and a linear function will always result in a graph with this shape? Explain your reasoning.

1

Talk the Talk

While you may not have realized it, the functions you worked with throughout this lesson are *polynomials*. A **polynomial** is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. The **degree** of a polynomial is the greatest variable exponent in the expression. For example, $4x^3 + 2x^2 + 5x + 1$ is a polynomial expression of degree three, $2x$ is a polynomial of degree 1, and a constant such as 5 has degree zero since it can be written as $5x^0$.

1. Given the functions,
   - $y_1 = ax^2$,
   - $y_2 = bx$, and
   - $y_3 = c$

   generalize the function family of the polynomial when:

   a. $y_1 + y_2$

   b. $y_1 + y_3$

   c. $y_2 + y_3$

2. When two functions of different degree are added, what can you say about the degree of the resulting function?

Be prepared to share your solutions and methods.